

Geometric Folding Algorithms: Linkages, Origami, Polyhedra

Updates to Chapter 22, Section 22.5: Special Classes of Edge-Unfoldable Polyhedra

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Abstract

Updates to Chapter 22, Edge-Unfolding of Polyhedra [DO07], specifically to Section 22.5, Special Classes of Edge-Unfoldable Polyhedra.

1 Introduction

We use the term *net* to mean a non-overlapping unfolding of a polyhedron \mathcal{P} to a polygon P . In particular, P avoids *strong overlap*, an overlap of two interior points. But *weak overlap*—boundary-to-boundary touching—is permitted (and thus P could be cut out of paper). So a net is a *weakly simple polygon* [CEX14]. This distinction between strong and weak overlap was not emphasized in *Geometric Folding Algorithms: Linkages, Origami, Polyhedra* (GFALOP).

Our focus throughout Chapter 22 is edge-unfolding to a net.

2 In GFALOP

The following infinite classes of polyhedra are discussed in Section 22.5:

- (1) Pyramids¹
- (2) Prisms
- (3) Prismoids
- (4) Domes
- (5) Convex generalized deltahedra²

¹Not explicitly discussed but a subset of domes.

²Originally called “higher-order” deltahedra. See [Bez07].

- (6) Gridded orthotubes [BDD⁺98]. (See discussion in Section 3.2.)

Note that prisms (for which edge-unfolding remains open) includes prisms which includes prisms.

3 Post GFALOP

The following additional classes of polyhedra have been proved to edge-unfold to nets.

3.1 Convex Polyhedra

- (7) Every convex polyhedron can be stretched via an affine transformation so that it has an edge-unfolding to a net [Gho14],
- (8) Nearly flat acutely triangulated convex cap, even if closed to a polyhedron by adding the convex polygonal base under the cap [O'R18] [O'R17].
- (9) Prisms: Tall or Rectangular Base [BDM21].
- (10) Polyhedra with a Hamiltonian simple closed quasigeodesic [O'R22]:
- (11) Polar zonohedra [O'R23].
- (12) Nested prisms [Rad24].
- (13) Tapered polyhedra; each is “skeletal” [OV24].

3.2 Gridded Orthogonal Polyhedra

Gridded orthogonal polyhedra allow cuts along the edges produced by slicing by coordinate planes through every vertex. Here we focus on unrefined grid-edge faces, i.e., a 1×1 grid. (The strongest result allowing grid-refinement is [DDFO17]: Every orthogonal polyhedron of genus ≤ 2 can be unfolded to a net with linear refinement.)

- (14) Orthogonal Terrains [O'R07].
- (15) Rectangle-faced orthostacks [CST12].
- (16) Orthotubes [DK22].
- (17) Grid-edge unfolding orthostacks with rectangular slabs [Per24].

3.3 Polycubes

A *polycube* is an orthogonal polyhedron built by face-to-face gluings of (unit) cubes. These are a special case of *gridded orthogonal polyhedra*, listed above.. It is natural to extend the notion of edge-unfolding to allow cuts along cube edges, which may not be edges of the corresponding orthogonal polyhedron. The main driver here is Open Problem 22.4, which remains open: Is there any genus-0 polycube that cannot be edge-unfolded (under the extended notion of edge-unfolding)? A negative result for special “edge zipper unfoldings” is [DDEO22].

- (18) Well-separated orthotrees [DFMO05].
- (19) One layer polycubes with sparse cubic holes [LPW14].
- (20) One layer polycubes with general cubic holes [Min22].
- (21) Polycubes with orthogonally convex layers [DM24].

References

- [BDD⁺98] Therese Biedl, Erik D. Demaine, Martin L. Demaine, Anna Lubiw, Joseph O’Rourke, Mark Overmars, Steve Robbins, and Sue Whitesides. Unfolding some classes of orthogonal polyhedra. In *Proc. 10th Canad. Conf. Comput. Geom.*, pages 70–71, 1998.
- [BDM21] Vincent Bian, Erik D. Demaine, and Rachana Madhukara. Edge-unfolding prisms: Tall or rectangular base. In *33rd Canad. Conf. Comput. Geom.*, 2021. arXiv 2106.14262.
- [Bez07] Daniel Bezdek. A proof of an extension of the icosahedral conjecture of steiner for generalized deltahedra. *Contributions Discrete Math.*, 2(1), 2007.
- [CEX14] Hsien-Chih Chang, Jeff Erickson, and Chao Xu. Detecting weakly simple polygons. In *Proc. 26th ACM-SIAM Symp. Discr. Algorithms (SODA)*, pages 1655–1670. SIAM, 2014.
- [CST12] Erin W. Chambers, Kyle Sykes, and Cynthia M. Traub. Unfolding rectangle-faced orthostacks. In *Canad. Conf. Comput. Geom.*, pages 23–28, 2012.
- [DDEO22] Erik D. Demaine, Martin L. Demaine, David Eppstein, and Joseph O’Rourke. Some polycubes have no edge-unzipping. *Geombinatorics*, XXXI:101–109, 2022.
- [DDFO17] Mirela Damian, Erik Demaine, Robin Flatland, and Joseph O’Rourke. Unfolding genus-2 orthogonal polyhedra with linear refinement. *Graphs and Combinatorics*, 33(5):1357–1379, 2017.
- [DFMO05] Mirela Damian, Robin Flatland, Henk Meijer, and Joseph O’Rourke. Unfolding well-separated orthotrees. In *15th Annu. Fall Workshop Comput. Geom.*, pages 23–25, November 2005.
- [DK22] Erik D Demaine and Kritkorn Karntikoon. Unfolding orthotubes with a dual hamiltonian path. *arXiv preprint arXiv:2201.12452*, 2022.
- [DM24] Mirela Damian and Henk Meijer. Edge-unfolding polycubes with orthogonally convex layers. In *Internat. Conf. Comb. Opt. Appl.*, pages 63–74. Springer, 2024. arXiv:2407.01326v2 [cs.CG].
- [DO07] Erik D. Demaine and Joseph O’Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra*. Cambridge University Press, 2007.
- [Gho14] Mohammad Ghomi. Affine unfoldings of convex polyhedra. *Geom. Topol.*, 18(5):3055–3090, 2014.

- [LPW14] Meng-Huan Liou, Sheung-Hung Poon, and Yu-Jie Wei. On edge-unfolding one-layer lattice polyhedra with cubic holes. In *Comput. Combinatorics: 20th Internat. Conf., COCOON*, pages 251–262. Springer, 2014.
- [Min22] Josef Minarík. Unfolding some classes of one-layer polycubes. In *Canad. Conf. Comput. Geom.*, pages 243–248, 2022.
- [O’R07] Joseph O’Rourke. Unfolding orthogonal terrains. Technical Report 084, Smith College, July 2007. arXiv:0707.0610v4 [cs.CG].
- [O’R17] Joseph O’Rourke. Addendum to: Edge-unfolding nearly flat convex caps. arXiv:1709.02433, 2017.
- [O’R18] Joseph O’Rourke. Edge-unfolding nearly flat convex caps. In *Proc. Symp. Comput. Geom. (SoCG)*, volume 99, pages 64:1–64:14. Leibniz Internat. Proc. Informatics, June 2018.
- [O’R22] Joseph O’Rourke. Hamiltonian quasigeodesics yield nets. arXiv:2206.05353v2 [cs.CG], 2022.
- [O’R23] Joseph O’Rourke. Polar zonohedra edge-unfold to nets. arXiv:2302.07747 [cs.CG], 2023.
- [OV24] Joseph O’Rourke and Costin Vîlcu. Skeletal cut loci on convex polyhedra. arXiv:2312.01534v3 [cs.CG], February 2024.
- [Per24] Klára Pernicová. Grid-edge unfolding orthostacks with rectangular slabs. In *Canad. Conf. Comput. Geom.*, pages 65–70, 2024.
- [Rad24] Manuel Radons. Edge-unfolding nested prisms. *Comput. Geom.*, 116:102033, 2024.